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# ON THE PROBLEM OF INSTABILITY OF THE BOUNDARY OF TWO MEDIA OF FINITE THICKNESS

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*A dispersion determinant nonlinear as far as the combination of frequency and relative velocity is concerned has been derived in the problem on propagation of waves at the boundary of liquid layers of finite thickness. The structure and physical meaning of the equation obtained have been discussed. The limiting cases have been considered.*

**Introduction.** The boundary of two phases is subjected to instabilities of different kinds, especially if one instability is exposed to external electromagnetic radiation. Earlier, in [1], a dispersion determinant combining the Frenkel–Tonks [2], Kelvin–Helmholtz, and Rayleigh–Taylor [3] instabilities was obtained in considering the problem on the behavior of small-amplitude waves on a charged horizontal interface of two liquids. The results obtained were appropriate for the case of two infinitely deep liquids; in the present work, conversely, an effort has been made to elucidate the manner in which these formulas are transformed in the general case of layers of bounded thickness. We note that a nontraditional approach to investigating this problem with introduction of the notion of effective surface tension has already been used in [4] and has led us to a number of new conclusions, although we considered quite a particular case.

**Derivation of the Dispersion Equation.** Let us find the dispersion equation of propagation of waves on a charged boundary of two immiscible incompressible liquids of different densities $\rho'$ and $\rho$, having thicknesses $h_1$ and $h_2$ of the upper and lower horizontal layers respectively; the upper liquid, an ideal dielectric, moves with a constant velocity $U$, whereas the lower liquid is viscous and conducting. Derivation of the dispersion equation is analogous to that in [1] and is carried out according to the procedure described in [5] in detail and subsequently developed in [6].

The unperturbed boundary is described by the equation $z = 0$. For the ideal liquid, we introduce the velocity potential

$$\Phi' = F \exp [i (kx - \omega t)] \cosh [k (h_1 - z)] + Ux . \tag{1}$$

On the interface, we can write

$$v_z = \frac{\partial \xi}{\partial t} , \tag{2}$$

$$v_z' = U \frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial t} . \tag{3}$$

For the lower liquid for an arbitrary viscosity, we seek the solution of the system of equations of motion (Navier–Stokes and continuity equations for small-amplitude waves)

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$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2}\right), \quad \frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2}\right) - g\,, \quad \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} = 0\,,$$

which is dependent on $t$ and $x$ as $\exp\,[i(kx - \omega t)]$ and is attenuated deep into the liquid ($z > 0$) in the form

$$v_x = \exp\,[i\,(kx - \omega t)]\,(A \sinh\,[k\,(z + h_2)] + B \sinh\,[l\,(z + h_2)])\,,$$

$$v_z = \exp\,[i\,(kx - \omega t)]\,(C \sinh\,[k\,(z + h_2)] + D \sinh\,[l\,(z + h_2)])\,, \tag{4}$$

$$P = \frac{\rho\omega}{k} A \exp\,[i\,(kx - \omega t)]\ \mathrm{ch}\,[k\,(z + h_2)] - \rho g z\,, \quad C = -iA\,, \quad D = -i\frac{k}{l}B\,, \quad l^2 = k^2 - \frac{i\omega}{\nu}\,.$$

Boundary conditions on the interface have the following form:
(1) tangential stresses are equal to zero: $\sigma_{xz} = 0$, i.e.,

$$\eta\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) = 0\,; \tag{5}$$

(2) the equality of pressures is

$$-P + 2\eta\frac{\partial v_z}{\partial z} - \gamma\frac{\partial^2 \xi}{\partial x^2} - 4\pi\sigma^2 k\xi = \rho'\frac{\partial \Phi'}{\partial t} + \rho' g\xi + \frac{\rho'}{2}(v'^2 - U^2)\,. \tag{6}$$

We substitute expressions (1) and (2) into (3), taking into account that $v'_z = \partial\Phi'/\partial z$, differentiate Eq. (3) with respect to time, and obtain, for the solution of the form (4),

$$C \sinh\,(kh_2) + D \sinh\,(lh_2) - \frac{k\omega \sinh\,(kh_1)}{Uk - \omega} F = 0\,. \tag{7}$$

From Eqs. (5), for the solutions of the form (4) we have

$$C + D\left(\frac{(1 - \chi)\cosh\,(lh_2) + \sinh\,(lh_2)}{\cosh\,(kh_2) + \sinh\,(kh_2)}\right) = 0\,, \quad \chi = \frac{i\omega}{\nu k^2}\,. \tag{8}$$

We differentiate the equality (6) with respect to time and substitute the expressions for $P$, $v_z$, and $v'^2 = v'^2_x + v'^2_z = \left(\frac{\partial \Phi'}{\partial x}\right)^2 + \left(\frac{\partial \Phi'}{\partial z}\right)^2$, after which we obtain

$$C \cosh\,(kh_2)\left(\frac{\chi}{2} + W(k) \tanh\,(kh_2) - 1\right) + D \sinh\,(lh_2)\left(W(k) - \frac{l}{k}\cosh\,(kh_2)\right)$$
$$+ F \cosh\,(kh_1)\,\frac{i\rho'}{\rho}\,\frac{Uk - \omega}{2\nu k} = 0\,, \tag{9}$$

$$W(k) = \frac{\gamma k^2 - 4\pi\sigma^2 k + (\rho - \rho')\,g}{2\eta i\omega k}\,.$$

We reduce Eqs. (7)–(9) into one system. The requirement of nontriviality of the solution for $C$, $D$, and $F$ yields the dispersion equation in the form of a dispersion determinant of 3rd order. After simple transformations, we will have

$$
\begin{vmatrix}
1 & 1 & 1 \\
\left\{\dfrac{\chi}{2}+W(k)\tanh(kh_2)-1\right\} & \left\{W(k)\tanh(kh_2)-\sqrt{1-\chi}\,\dfrac{\tanh(kh_2)}{\tanh(lh_2)}\right\} & \alpha\,\dfrac{(Uk-\omega)^2}{2\nu^2k^4\chi}\,\dfrac{\tanh(kh_2)}{\tanh(kh_1)} \\
1 & \dfrac{(1-\chi)\cosh(lh_2)+1}{\cosh(kh_2)+1} & 0
\end{vmatrix} = 0\,. \tag{10}
$$

It is Eq. (10) that is the dispersion equation sought, which is written in implicit form; the third line has already been divided by $\eta \neq 0$.

We expand the dispersion determinant (10) and, after elementary transformations, have

$$
(2-\chi)(1+\tanh(lh_2)-\chi)+(\Omega(k)+K(k,\omega))\,\frac{(\chi-1)+\dfrac{\tanh(lh_2)}{\tanh(kh_2)}}{\chi} = 2(1+\tanh(kh_2))\sqrt{1-\chi}\,, \tag{11}
$$

where

$$
K(k,\omega) = -\alpha\,\frac{(Uk-\omega)^2}{\nu^2k^4}\,\frac{\tanh(kh_2)}{\tanh(kh_1)}\,;\quad \Omega(k) = \frac{\gamma k^2-4\pi\sigma^2k+(1-\alpha)\rho g}{\rho\nu^2k^3}\tanh(kh_2)\,;\quad \alpha = \frac{\rho'}{\rho}\,.
$$

We note that the dispersion equation obtained is substantially nonlinear in character compared to the analogous equation derived earlier for liquids [1] occupying the upper and lower half-spaces respectively.

**Analysis of the Equation Obtained.** *Case of Two Infinitely Deep Liquids.* For the limiting transition $h_1 \to \infty$ and $h_2 \to \infty$, Eq. (11) is strongly simplified and takes the form

$$
(2-\chi)^2+\Omega(k)+K(k,\omega) = 4\sqrt{1-\chi}\,. \tag{12}
$$

Let us numerically analyze its solutions in dimensionless variables. For this purpose, we introduce the dimensionless variable of velocity $\theta = U/\nu k$. Then Eq. (12) allows representation in the form

$$
(2-\chi)^2+\Omega(k)-\alpha(\theta+i\chi)^2 = 4\sqrt{1-\chi}\,.
$$

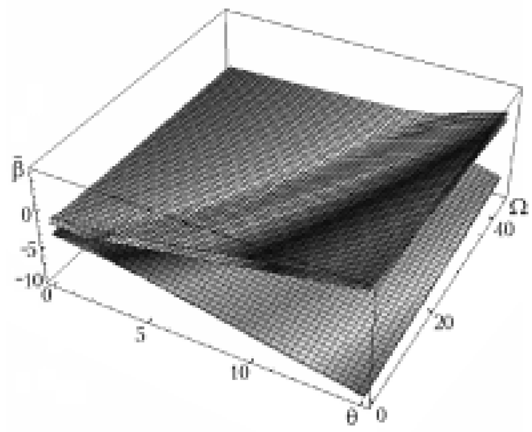
We introduce the notation $\omega = \omega' + i\beta$. The real and imaginary parts of the solutions for $\chi$, the dimensionless attenuation coefficient, and the dimensionless cyclic frequency are related by the relations

$$
\mathrm{Re}(\chi) = -\frac{\beta}{\nu k^2} = -\tilde{\beta}\,,\quad \mathrm{Im}(\chi) = \frac{\omega'}{\nu k^2} = \tilde{\omega}\,. \tag{13}
$$

In solving Eq. (12) numerically, we obtain dispersion surfaces reflecting the dependence of the dimensionless frequency and attenuation coefficient (13) on the dimensionless velocity of the upper-liquid flow $\theta$ and the function $\Omega$. The negative values of the dimensionless frequency correspond to the phase shift of the running wave by $\pi$. The negative values of the dimensionless attenuation coefficient correspond to the amplitude attenuation of the wave, whereas the positive values correspond to the amplitude growth in instability. The dispersion surfaces of the dimensionless attenuation coefficient are represented in Fig. 1.

Of interest is the form of the set of points of the stable-to-unstable transition of the wave, i.e., the separation of the wave, which corresponds to the set of points with a zero dimensionless attenuation coefficient. Imposition of the condition $\mathrm{Re}[\chi] = 0$ on the solution of Eq. (12) produces a pair of C-shaped curves (we call them C curves) in the $\theta$–$\Omega$ plane (Fig. 2).

The peak of the left-hand C curve is located at the point ($\Omega = -8$, $\theta = 0$), which confirms the result obtained in [6]; the peak of the right-hand C curve passes through the origin of coordinates. The C curves have no common



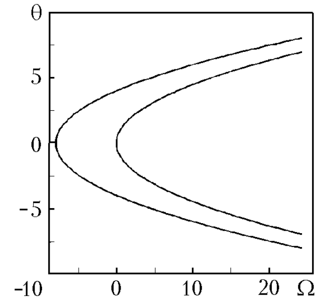




Fig. 1. Dispersion surfaces of the dimensionless attenuation coefficient, $\alpha = 1$.

Fig. 2. C curves, $\alpha = 1$.

points. When $\Omega \to \infty$ the C curves join asymptotically, describing, in the limit, either an ideal liquid or long waves for which neglect of viscosity holds. The domain to the left of the left-hand C curve is the domain of existence of two unstable waves, whereas the domain to the right of the right-hand C curve is the domain of stable waves; the domain between the two C curves is transient, in which stable and unstable waves simultaneously exist. When the upper liquid is absent, the C curves degenerate into vertical straight lines passing through the indicated points of the peaks.

*Case of Two Ideal Liquids.* Let us consider the case where the viscosity of the lower liquid is negligible. The corresponding dispersion relation is obtained for the limiting transition $\nu \to 0$. As a result, we have

$$\omega^2 - 2b_1\omega - b_0 = 0\,, \quad b_1 = \frac{\alpha U}{1+\alpha} k \tanh(kh_2)\,, \quad b_0 = \frac{\gamma k^2 - (4\pi\sigma^2 - \alpha\rho U^2)\,k + (1-\alpha)\,\rho g}{(1+\alpha)\,\rho} k \tanh(kh_2)\,.$$

The solution has two modes and allows representation in explicit form:

$$\omega_{1,2} = b_1 \pm \sqrt{b_1^2 + b_0}\,. \tag{14}$$

We expand expression (13) in a series in $k$ in the vicinity of $k = 0$ to a term of the order of $k^3$ (for the sake of simplicity we consider just the first mode; for the second mode, the reasoning is analogous):

$$\omega = c_0 k + c_1 k^2 + c_2 k^3\,, \tag{15}$$

where

$$c_0 = \sqrt{\frac{1-\alpha}{1+\alpha} g h_2}\;; \quad c_1 = \frac{\alpha h_2}{2\,(1+\alpha)\,c_0}\left(U^2 + 2c_0 U - \frac{4\pi\sigma^2}{\alpha\rho}\right);$$

$$c_2 = \frac{1}{2c_0\,(1+\alpha)}\left(\left(\alpha U^2 h_2^2 + (1+\alpha)\,\frac{\gamma h_2 - \frac{1}{3}\rho g h_2^3\,(1-\alpha)}{\rho}\right) + \frac{1}{4c_0^3}\left(\alpha U^2 - \frac{4\pi\sigma^2}{\rho}\right)^2 h_2^2\right).$$

The coefficient $c_1$ characterizes the dependence of the phase velocity of propagation of the wave on the value of the wave number $k$:

$$c_0 = \left.\frac{\partial\omega}{\partial k}\right|_{k=0}\,, \quad c_1 = \frac{1}{2}\left.\frac{\partial^2\omega}{\partial k^2}\right|_{k=0} = \frac{1}{2}\left.\frac{\partial c_0}{\partial k}\right|_{k=0}\,.$$

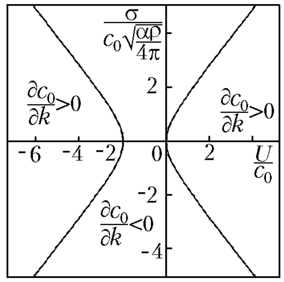



Fig. 3. Dispersion hyperbolas.

It is noteworthy that the appearance of a term quadratic in $k$ in the expansion of $\omega$ is not accidental and is due to the presence of the separated direction of propagation of the wave ($U \neq 0$) and to the appearance of an electric field.

When $U = 0$ we invariably have $c_1 < 0$, i.e., the phase velocity of propagation of waves grows as the wavelength decreases (anomalous dispersion). When $\sigma = 0$, conversely, we have $c_1 > 0$, i.e., the phase wave velocity grows with wavelength (normal dispersion).

From the above reasoning, we can introduce a geometric interpretation of a set of states characterizing the transition between the normal and anomalous dispersions — dispersion hyperbolas (Fig. 3):

$$\left(\frac{U + c_0}{c_0}\right)^2 - \left(\frac{\sigma}{c_0 \sqrt{\alpha\rho/4\pi}}\right)^2 = 1 \, , \quad c_1 = 0 \, .$$

The domain between two branches of the dispersion hyperbolas correspond to states for which we have $\partial c_0/\partial k < 0$; in the remaining domains, we have $\partial c_0/\partial k > 0$.

*Case of a "Shallow" Viscous Liquid.* The case of a thin layer of a viscous liquid and of a deep ideal liquid is of interest. Such a situation can occur when liquid viscous films are considered.

After corresponding simplifications, Eq. (11) will take the form

$$(1 + p^2)\,(\psi + p)\,(1 - p) + \psi\,(\Omega\,(k) + K\,(k, \omega)) = 2\,(1 + \psi)\,(1 - p)\, ,$$

$$\psi = kh_2\, , \quad p = \sqrt{1 - \chi}\, , \quad K\,(k, \omega) = -\,\alpha\,\frac{(Uk - \omega)^2}{\nu^2 k^4}\, , \quad \Omega\,(k) = \frac{\gamma k^2 - 4\pi\sigma^2 k + (1 - \alpha)\,\rho g}{\rho \nu^2 k^3}\tanh(kh_2)\, .$$

When $\psi \to 0$ (waves on the film surface) we obtain a degenerate dispersion equation:

$$p\,(1 + p^2)\,(1 - p) = 2\,(1 - p)\, , \quad p_{1,2} = 1\, , \quad p_{3,4} = -\frac{1}{2} \pm i\,\frac{\sqrt{7}}{2}\, ,$$

whence we have

$$\omega_{1,2} = 0\, , \quad \omega_{3,4} = \left(\pm\frac{\sqrt{7}}{2} - \frac{5}{2}\,i\right)\nu k^2\, .$$

The dispersion surfaces degenerate into three parallel planes. A completely different situation is observed when the ideal liquid is not infinitely deep. The dispersion relation will take the form

$$(1 + p^2)\,(\psi + p)\,(1 - p) + \psi\left(\Omega\,(k) + \frac{K\,(k, \omega)}{\psi}\,\frac{h_2}{h_1}\right) = 2\,(1 + \psi)\,(1 - p)\, .$$

When $\psi \to 0$ we obtain the degenerate dispersion equation

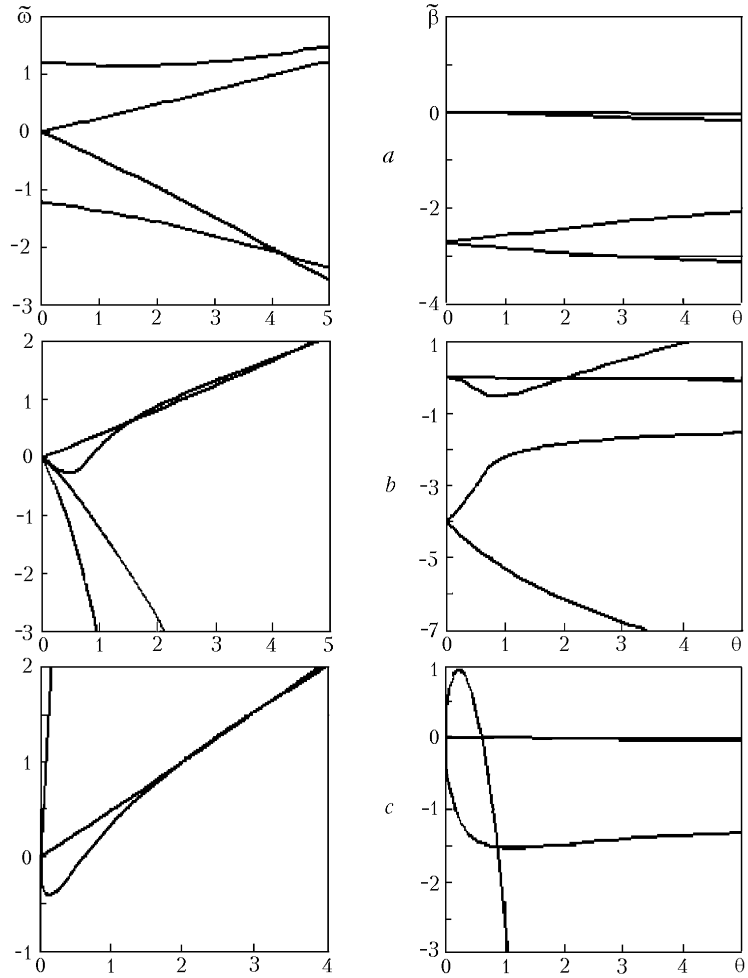



Fig. 4. Dimensionless frequency and attenuation coefficient vs. dimensionless velocity: a) $\alpha h_2/h_1$ = 0.1, b) 0.5, and c) 1.0.

$$(1+p^2)(1-p)\,p+\frac{\alpha h_2}{h_1}(i\theta+p^2-1)^2=2(1-p)\,. \tag{16}$$

Since it is difficult to analytically solve the equation obtained, we made a numerical analysis. Figure 4 plots the solutions for different values of the parameters $\alpha h_2/\alpha h_1$ and θ; the plots quite clearly reflect all special features of the phenomenon in question.

## NOTATION

$A$, $B$, certain constants, m/sec; $b_0$, coefficient, sec$^{-1}$; $b_1$, coefficient, sec$^{-2}$; $C$, constant, m/sec; $c_0$, phase velocity of waves, m/sec; $C_1$, dispersion coefficient, m$^2$/sec; $c_2$, dispersion parameter, m$^3$/sec; $D$, constant, m/sec; $g$, free-fall acceleration, m/sec$^2$; $F$, constant, m$^2$/sec; $h_1$ and $h_2$, thicknesses of the upper and lower liquids, m; $i$, imaginary unit; $k$, wave-vector modulus, m$^{-1}$; $l$, function of the wave-vector modulus, m$^{-1}$; $p$, function of the complex frequency; $P$, pressure, Pa; $t$, time, sec; $U$, horizontal velocity of the upper liquid at infinity, m/sec; $v'$, vector velocity of the upper liquid, m/sec; $v_x$ and $v_z$, horizontal and vertical components of the velocity of the lower liquid, m/sec; $v'_x$ and $v'_z$, horizontal and vertical components of the velocity of the upper liquid, m/sec; $W$, function of the wave-vector modulus; $x$, horizontal axis; $z$, vertical axis; α, relative-density coefficient; β, attenuation coefficient, sec$^{-1}$; $\tilde{\beta}$, dimensionless attenuation coefficient; γ, specific surface energy, J/m$^2$; η, dynamic viscosity, Pa·sec; θ, dimensionless velocity; $\xi$, vertical coordinate of the interface, m; $\rho$ and $\rho'$, densities of the lower and upper liquid, $kg/m^3$; $\sigma$, charge density on the interface, $C/m^2$; $\sigma_{xz}$, component of the stress tensor, Pa; $\nu$, kinematic viscosity, $m^2/sec$; $\Phi'$, velocity potential, $m^2/sec$; $\chi$, dimensionless parameter of complex frequency; $\psi$, parameter of the liquid's depth; $\omega$, complex frequency, $sec^{-1}$; $\Omega$, function of the wave-vector modulus; $\omega'$, cyclic frequency, $sec^{-1}$; $\tilde{\omega}$, dimensionless cyclic frequency. Subscripts: $x$, projection onto the $0x$ axis; $z$, projection onto the $0z$ axis.

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